Section 3.5

1. (a) $2x + 2y = 200$ so $y = 100 - x$. Maximize $A = x \cdot y = x \cdot (100 - x) = 100x - x^2$.
$A' = 100 - 2x$ and $A' = 0$ when $x = 50$ ($y = 100 - x = 50$). $A'' = -2 < 0$ so $x = 50$ yields the maximum enclosed area. When $x = 50$,
$A = 50(100 - 50) = 2500$ square feet.

(b) $2x + 2y = P$ so $y = P/2 - x$. Maximize $A = x \cdot y = x \cdot (P/2 - x) = (P/2)x - x^2$.
$A' = P/2 - 2x$ and $A' = 0$ when $x = P/4$ (then $y = P/2 - x = P/4$). $A'' = -2 < 0$ so $x = P/4$ yields the maximum enclosed area.
This garden is a $P/4$ by $P/4$ square.

(c) $2x + y = P$ so $y = P - 2x$. Maximize $A = xy = x(P - 2x) = Px - 2x^2$.
$A' = P - 4x$ and $A' = 0$ when $x = P/4$ (then $y = P - 2x = P/2$).

(d) A circle. A semicircle.

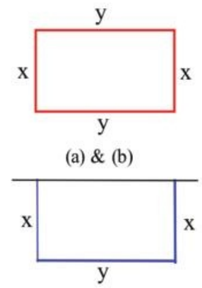
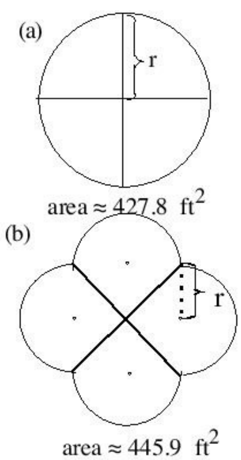


Fig. 3.5P1

3. (a) $120 = 2x + 5y$ so $y = 24 - \frac{2}{5}x$. Maximize $A = xy = x(24 - \frac{2}{5} x) = 24x - \frac{2}{5} x^2$.
$A' = 24 - \frac{4}{5} x$ and $A' = 0$ when $x = 30$ (then $y = 12$). $A'' = -4/5 < 0$ so $x = 30$ yields the maximum enclosed area. Area is (30 ft)(12 ft) = 360 square feet.

(b) A circular pen divided into 4 equal stalls by two diameters shown in diagram (a) does a better job than a square with 400 square feet. If the radius is r, then $4r + 2\pi r = 120$ so $r = 120/(4 + 2\pi) \approx 11.67$.
The resulting enclosed area is $A = \pi r^2 \approx \pi(11.67)^2 \approx 427.8$ sq. ft.

The pen shown in diagram (b) does even better. If each semicircle has radius r, then the figure uses $4\sqrt{2}\ r + 4\pi r = 120$ feet of fence so $r = 120/(4\sqrt{2} + 4\pi) \approx 6.585$. The resulting enclosed area is
$A = \text{(square)} + \text{(four semicircles)} = (2r)^2 + 4(\frac{1}{2} \pi r^2) \approx 445.90$ sq. ft.



5. $2x + 2y = 10$ so $y = 5 - x$.
Maximize $V = xy(10 - 2x) = x(5 - x)(10 - 2x) = 50x - 20x^2 + 2x^3$.
$V' = 50 - 40x + 6x^2 = 2(3x - 5)(x - 5)$ and $V' = 0$ when $x = 5$ and $x = 5/3$. When $x = 5$, then $V = 0$, clearly not a maximum, so $x = 5/3$. The dimensions of the box with the largest volume are 5/3, 10/3, and 20/3.

7. (a) $V = \pi r^2 h = 100$ so $h = \frac{100}{\pi r^2}$.
Minimize $C = 2\text{(top area)} + 5\text{(bottom area)} + 3\text{(side area)}$
$= 2(\pi r^2) + 5(\pi r^2) + 3(2\pi rh) = 7\pi r^2 + 6\pi r (\frac{100}{\pi r^2}) = 7\pi r^2 + \frac{600}{r}$.

$C' = 14\pi r - \frac{600}{r^2}$ and $C' = 0$ when $r = \sqrt[3]{600/(14\pi)} \approx 2.39$ (then $h = \frac{100}{\pi r^2} \approx 5.57$).

(b) Let k = top + bottom rate = 2¢ + the bottom rate > 2¢ + 5¢ = 7¢. Minimize $C = k\pi r^2 + \frac{600}{r}$.

$C' = 2k\pi r - \frac{600}{r^2}$ and $C' = 0$ when $r = \sqrt[3]{600/(2k\pi)}$. If $k = 8$, then $r \approx 2.29$.
If $k = 9$, then $r \approx 2.20$. If $k = 10$, then $r \approx 2.12$. As the cost of the bottom material increases, the radius of the least expensive cylindrical can decreases: the least expensive can becomes narrower and taller

9. Time = distance/rate. Run distance = x ($0 \le x \le 60$ Why?) so run time = x/8.

Swim distance = $\sqrt{40^2 + (60-x)^2}$ so swim time = $\frac{1}{2}\sqrt{40^2 + (60-x)^2}$ and the total time is

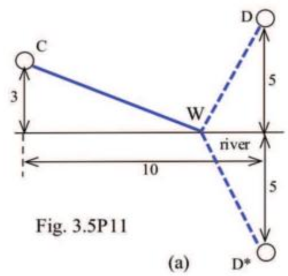
$$T = \frac{x}{8} + \frac{1}{2}\sqrt{40^2 + (60-x)^2} \ .$$

$$T' = \frac{1}{8} + \frac{1}{2}\frac{1}{2}(40^2 + (60-x)^2)^{-1/2} \cdot 2 \cdot (60-x) \cdot (-1) = \frac{1}{8} - \frac{60-x}{2\sqrt{40^2 + (60-x)^2}} \ .$$

$T' = 0$ when $x = 60 \pm \frac{40}{\sqrt{15}}$. The value $x = 60 + \frac{40}{\sqrt{15}} > 60$ so the least total time occurs when

$x = 60 - \frac{40}{\sqrt{15}} \approx 49.7$ meters. In this situation, the lifeguard should run about 5/6 of the way along the beach before going into the water.

11. (a) Consider a similar problem with a new town D* located at the "mirror image" of D across the river (Fig. 3.5P11a). If the water works is built at any location W along the river, then the distances are the same from W to D and to D*: dist(W,D) = dist(W,D*).
Then dist(C,W) + dist(W,D) = dist(C,W) + dist(W,D*). The shortest distance from C to D* is a straight line (Fig. 3.5P11b), and this straight line gives similar triangles with equal side ratios: $\frac{x}{3} = \frac{10-x}{5}$ so $x = 15/4 = 3.75$ miles. A consequence of this "mirror image" view of the problem is that "at the best location W the angle of incidence α equals the angle of reflection β "



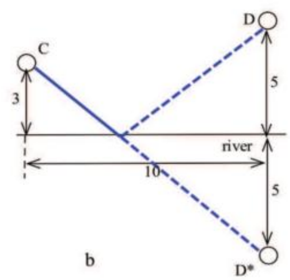


Fig. 3.5P11

(b) Minimize $C = 3000\text{dist}(C,W) + 7000\text{dist}(W,D) = 3000\sqrt{x^2+9} + 7000\sqrt{(10-x)^2+25}$.

$$C' = \frac{3000x}{\sqrt{x^2+9}} + \frac{-7000(10-x)}{\sqrt{(10-x)^2+25}} \quad \text{so}$$

$$C' = 0 \text{ when } \frac{3x}{\sqrt{x^2+9}} = \frac{7(10-x)}{\sqrt{(10-x)^2+25}} \quad \text{and } x \approx 7.82 \text{ miles.}$$

As it becomes relatively more expensive to build the pipe from a point W on the river to D, the cheapest route tends to shorten the distance from W to D.

13. (a) Let x be the length of one edge of the square end. Then $V = x^2(108 - 4x) = 108x^2 - 4x^3$.
$V' = 216x - 12x^2 = 6x(18 - x)$ so $V' = 0$ when $x = 0$ or $x = 18$. The dimensions of the greatest volume acceptable box with a square end are 18 by 18 by 36 inches: $V = 11{,}664 \text{ in}^3$.

(b) Let x be the length of the shorter edge of the end. Then $V = 2x^2(108 - 6x) = 216x^2 - 12x^3$.
$V' = 432x - 36x^2 = 36x(12 - x)$ so $V' = 0$ when $x = 0$ or $x = 12$. The dimensions of the largest box acceptable box with this shape are 12 by 24 by 36 inches: $V = 10{,}368 \text{ in}^3$.

(c) Let x be the radius of the circular end. Then $V = \pi x^2(108 - 2\pi x) = 108\pi x^2 - 2\pi^2 x^3$.
$V' = 216\pi x - 6\pi^2 x^2 = 6\pi x(36 - \pi x)$ so $V' = 0$ when $x = 0$ or $x = 36/\pi \approx 11.46$ inches. The dimensions of the largest box acceptable box with a circular end are a radius of $36/\pi \approx 11.46$ and a length of 36 inches: $V \approx 14{,}851 \text{ in}^3$.

21. $V = \frac{1}{3}\pi r^2 h$ and $h = \sqrt{9 - r^2}$ so $V = \frac{1}{3}\pi r^2 \sqrt{9 - r^2} = \frac{\pi}{3}\sqrt{9r^4 - r^6}$. Then

$V' = \frac{\pi}{3} \frac{36r^3 - 6r^5}{\sqrt{9r^4 - r^6}}$, and $V' = 0$ when $36r^3 = 6r^5$ so $r = \sqrt{6} \approx 2.45$ inches and

$h = \sqrt{9 - r^2} = \sqrt{3} \approx 1.73$ inches.

31. (a) $y = 20 - \frac{20}{50}x$. $A = (\text{base})\text{height}) = xy = x(20 - \frac{2}{5}x) = 20x - \frac{2}{5}x^2$.

$A' = 20 - \frac{4}{5}x = 0$ when $x = 25$. Then $y = 10$ and Area $= 250$.

(b) $y = H - \frac{H}{B}x$. $A = (\text{base})\text{height}) = x(H - \frac{H}{B}x) = Hx - \frac{H}{B}x^2$.

$A' = H - \frac{2H}{B}x = 0$ when $x = \frac{B}{2}$. Then $y = \frac{H}{2}$ and Area $= \frac{BH}{4}$.