4.3: Rational Functions:

Rational Function:

$f(x) = \frac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials, and $h(x) \neq 0$

Note: $h(x)$ can be a constant

Example:

$f(x) = \frac{3x+5x^2}{\sqrt{3}}, = \frac{3x^3-2x^2}{3x^2}$, $g(x) = 3x^3 - 2x^2$ are all rational functions

$y = \frac{3x+5x^2}{\sqrt{3x}}, f(x) = \frac{3x+5x^{1.5}}{\sqrt{3}}$ are not rational functions

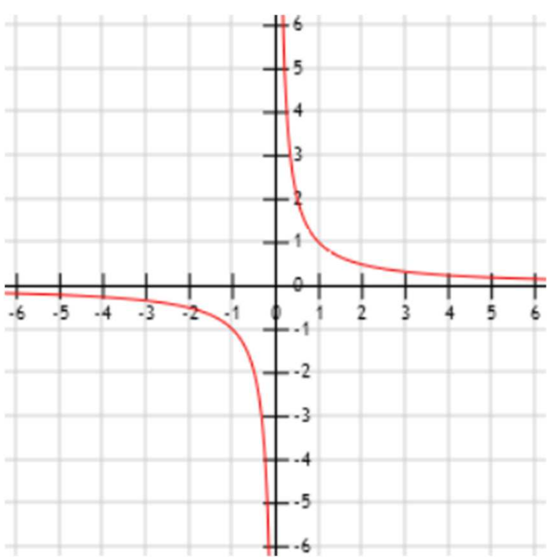
Asymptotes:

An asymptote is a line that part of the graph approaches.

Reminder:

For the function:

$$y = \frac{1}{x}$$



The graph appears to get close to the lines y = 0 , and x = 0.

Vertical Asymptotes of Rational Functions:

Given $f(x) = \frac{g(x)}{h(x)}$ be a simplified rational function. If *c* is a zero of *h(x)*, then the line *x=c* is a vertical asymptote.

Example 1: Find the vertical asymptotes

a.) $f(x) = \frac{x}{x^2-5x-6}$

b.) $g(x) = \frac{4-x}{x^4-81x^2}$

Horizontal Asymptotes of Rational Functions:

Given $f(x) = \frac{g(x)}{h(x)}$ and m is the degree of the numerator (top) and n is the degree of the denominator (bottom) . In other words

$$\frac{g(x)}{h(x)} = \frac{a_m x^m + \cdots + a_0}{b_n x^n + \cdots + b_0}$$

1. If $m < n$, then y = 0 is a horizontal asymptote.

2. If $m = n$, then $y = \frac{a_m}{b_n}$ (ratio of the leading coefficients) is the horizontal asymptote.

3. If $m > n$, then there is a slant or oblique asymptote (which is not covered in this course).

Alternatively, examining the end behaviour by letting
x -> ∞ and x -> -∞ (or substituting x with a relatively large value) we can determine the horizontal asymptote (if it exists).

Example 2: Determine the horizontal asymptote of the rational functions:

a.) $f(x) = \frac{4x-6}{3x^2+4x-2}$

b.) $g(x) = \frac{6-x^2}{3x^2-4x+4}$

c.) $h(x) = \frac{x^3-3x-2}{3x^2}$

Holes in Rational Functions:

A hole (or discontinuity) exists in a rational function if a value of *x* sets both the numerator and denominator to 0.

Consider the rational function:

$$f(x) = \frac{x-3}{(x-3)(x+2)}$$

At first, it appears there are vertical asymptotes at *x* = -2 and 3, but when the function is simplified and reduced:

$f(x) = \frac{1}{x+2}$ (*note: x +2 stays in the denominator)

From the simplified form, there is one vertical asymptote at *x* = -2, the domain $x \neq 3$ still exists from the original function and thus a hole is created at x = 3.

To find the y value of the hole substitute the x value into the simplified function: $f(3) =$

The point (3, $\frac{1}{5}$) is a hole.

Sketch of $f(x) = \frac{x-3}{(x-3)(x+2)}$

Example 3: Determine the hole of the rational function and sketch a graph:

$$f(x) = \frac{x^2 - 9}{x + 3}$$

x-intercept(s) and y-intercept of Rational Functions

The x-intercept is when y = 0. In the case of rational functions, we can find the x-values when the numerator is equal to 0 (and exists on the domain)

The y-intercept is when x = 0. Substitute x=0 and find the y=value. There is only 1 y-intercept.

Example 4: Determine the x- and y- intercepts of the following rational functions:

a.) $f(x) = \frac{(x-2)(x+3)(x-4)}{(x+1)(x-1)}$

b.) $g(x) = \frac{x^3-5x^2-14x}{4x^2-9}$

c.) $h(x) = \frac{2x^2+12x+18}{x^2-9}$